TRANSFORMING PRE-SERVICE TEACHERS DEFINITION OF FUNCTION

Milan F Sherman
Drake University
milan.sherman@drake.edu

Michael S Meagher
Brooklyn College - CUNY
mmeagher@brooklyn.cuny.edu

Jennifer Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Allison McCulloch
UNC Charlotte
amccul11@uncc.edu

Functions form a central part of the U.S. mathematics curriculum especially at the high school level. There is a considerable body of research showing that students at all levels, including preservice secondary mathematics teachers, have difficulties with the definition of function as a correspondence between two sets with a univalence condition. Those difficulties include privileging algebraic representations and reductive interpretations of the univalence condition in the form of the vertical line test. In our research study, 47 pre-service mathematics teachers provided definitions of function, engaged with an interactive applet that had a non-standard representation of function, and then provided revised definitions of function. The results of the study show a measurable increase in the participants level of abstraction in their definitions, and an increase in their attention to the univalence condition.

Keywords: Teacher Education - Preservice, Teacher Knowledge, Technology

Introduction

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Students are provided experiences with functions from the very earliest grades, usually pattern exploration, up to and through high school with a formal treatment of functions as arbitrary mappings between sets. Indeed, in the Common Core State Standards for Mathematics the study of function is given its own domain in grades 9-12 (National Governors Association for Best Practices & Council of Chief State School Officers, 2010).

There is an extensive body of research on students' understanding of function (e.g., Carlson et al., 2003; Cooney et al., 2010; Dubinsky & Harel, 1992; Even, 1990; 1993, Oehrtman et al., 2008) and much of that research reports that learners (secondary, post-secondary as well as pre-service and in-service teachers) have considerable difficulty identifying functions and in distinguishing them from non-functions.

Given the emphasis on function in school mathematics, it is important to consider preservice secondary mathematics teachers' (PSMTs) conceptions of function. Since PSMTs must possess robust conceptions of function so they can plan for supporting the development of their future students' function understandings. As such, mathematics teacher educators need to identify methods for eliciting and transforming PSMTs' conceptions of function to meet these needs. The decades of research outlining the details of both students' and teachers' flawed and limited conceptions of function provide grounding for thinking about how to support PSMTs' further conception development. Coupling this vast literature and transformation theory (Mezirow, 2000; Taylor, 2007), a specifically adult constructivist learning theory, we designed a task utilizing advanced digital technology to meet this need. The purpose of this study is to examine the ways in which this task elicited and transformed PSMTs' personal definitions of function. In doing so we aim to add to the knowledge base of designing learning experiences for PSMTs that problematize and support transformation of important mathematical conceptions.

Literature Review and Relationship to Research

Defining Function

In Thompson & Carlson's (2017) discussion of the evolution of the definition of function in the history of mathematics, they describe how a variation and covariation conception of function came to be replaced, owing to the emerging dominance of a set theoretic conception of variable as used in group theory and other areas, by a correspondence conception of function that "solved problems that arose for mathematicians, [but that] introducing it in school mathematics made it nearly impossible for school students to see any intellectual need for it" (p.422). This abstract correspondence definition is often referred to as the Dirichlet-Bourbaki definition of function and states that a function is a correspondence between arbitrary sets satisfying a univalence condition i.e. each element in the domain corresponds to exactly one element in the codomain.

Thompson and Carlson (2017), citing Cooney and Wilson (1993), as well as drawing on their own review of 17 U.S. Precalculus textbooks, note that a correspondence definition of function is used exclusively in all of these textbooks. Therefore, while we expect that most students (and PSMTs) who have attended U.S. schools to have experience with a definition involving a correspondence (or mapping) between two sets with constraints on the mapping of individual elements (the univalence condition), Even (1993) notes that many students retain a "protypic" (p.96) concept of functions as linear relationships and "many expect graphs of functions to be "reasonable" and functions to be representable by a formula." (p. 96).

Teachers' Understandings of the Function Concept

In addition to content knowledge of functions, mathematics teachers require Mathematical Knowledge for Teaching (MKT) (Ball, Hill, & Bass, 2005) of functions in order to be effective, i.e. teachers should be aware of various representations of functions, many examples of functions and non-functions, and known areas of challenge for students when learning functions. However, the situation regarding teachers' understanding of function, at the content level, is quite similar to that of school and college students (Bannister, 2014; Even, 1990, 1993; Wilson, 1994). In particular, similar to students, practicing teachers and PSMTs tend to privilege algebraic representations of functions and emphasise properties of graphs (e.g., vertical line test) in their descriptions of functions and non-functions (Even, 1990, 1993; Wilson, 1994). They also exhibit a limited repertoire of representations on which to draw in helping students understand functions (Bannister, 2014; Hatisaru & Erbas, 2017).

Crucially, teachers' understanding of function has been shown to impact the pedagogical choices they make during instruction. In a study of 152 PSMTs, Even (1993) found they could not justify the need for univalence and did not know why it was important to distinguish between functions and non-functions. Owing to this lack of content knowledge, the PSMTs' MKT was constrained and they limited the exposure of their students to various function representations and emphasised procedures such as the vertical line test in identifying functions.

Learning in Technology-rich Environments

The importance of advanced digital technologies in mathematics education is now well established. Major stakeholders such as The National Council of Teachers of Mathematics (NCTM) have asserted that "It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication" (NCTM, 2015, p.1). A considerable body of research supports the idea that advanced digital technologies can support learning in general (Tamim et al., 2011) and mathematics concepts in particular (Drijvers et al., 2010; Olive et al., 2010.)

In explaining the importance of design Drijvers (2015) argues that "the criterion for appropriate design is that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake." (p.15). For example, a good design to allow students to engage with the concept of function will allow the user to experience different kinds of functions and non-functions with enough data to differentiate between the two.

Theoretical Framework

Transformation Theory

Given the preponderance of evidence in the literature that the conception of function of PSMTs is often underdeveloped, our goal was to design a learning experience that problematized those conceptions, required PSMTs to reflect on them, and, ideally, resulted in further development and refinement of their conception as articulated in their personal definition (i.e., learning). Given that PSMTs come to their methods courses as adults and with a wealth of previous experiences related to the function concept, we turned to an adult learning theory, namely Mezirow's (2000) transformation theory. Transformation theory is an adult learning theory that is consistent with constructivist assumptions and expands on those assumptions by acknowledging the broad predispositions an adult might have toward a concept based on prior experiences, and the role these dispositions play in their meaning making (Mezirow, 2000).

Mezirow (2009) describes four forms of learning at the heart of transformation theory: elaborating existing meaning schemes, creating new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. According to Mezirow (2009), learning by transforming existing meaning schemes and perspectives often begins with a stimulus, a *disorienting dilemma*, which requires one to question their current meaning schemes. However, experiencing a disorienting dilemma alone is not enough to effect a transformation and learning will only occur through critical reflection (Merriam, 2004; Mezirow, 2000; Taylor, 2007).

A Transformative Learning Experience for the Concept of Function

There is a significant research base that recognizes that PSMTs often have conceptions of function that are inconsistent with the concept itself (e.g., Bakar & Tall, 1991; Breidenbach, et al., 1992; Carlson, 1998; Carlson & Oehrtman, 2005; Rasmussen, 2000; Vinner & Dreyfus, 1989). Given that PSMTs will be responsible for teaching others about function, it is important to try to address this concern through carefully designed learning experiences.

The idea of a *cognitive root* was introduced by Tall et al. (2000) as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" as they were developing a cognitive approach to calculus (Tall et al., 2000, p.497). As an example of a cognitive root for function concepts, Tall et al. suggest the use of a *function machine* (sometimes referred to as a function box). The machine metaphor Tall and colleagues describe is typically a "guess my rule" activity which is algebraic in nature. Studies using function machines were promising but some students still struggled with connecting different representations and determining what is and is not a function (e.g. McGowen et al., 2000).

Given the promise of cognitive roots we set out to design a machine-based experience using representations that were unfamiliar for PSMTs as a stimulus for examining their meaning schemes of function. The applet we designed, built on the metaphor of a vending machine, contained no numerical or algebraic expressions. Our intention was to put PSMTs in a context in which they would not be able to automatically rely on an algebraic, and often procedural, conceptions of functions (e.g., use of the vertical line test).

The Vending Machine applet (https://ggbm.at/X3Cn7npQ) consists of four pages; each with two to six vending machines and asks the user to identify each vending machine as a function or non-function (Figure 1). The machines each consist of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is pressed it produces none, one, or more than one of the different colored cans which may, or may not, correspond to the color of the button pressed.

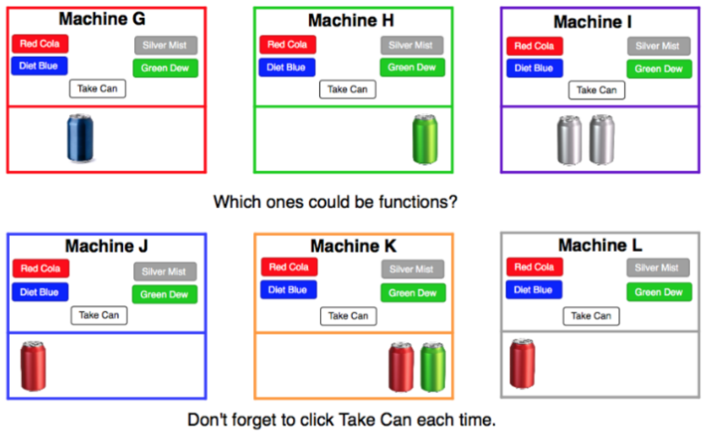

Figure 1: Vending Machine Applet

By removing numeric and algebraic representations, the applet could allow PSMTs to attend to the nature of input and outputs and the relationship between them. We intentionally designed to trigger dilemmas related to known issues from the literature e.g. researchers have shown that students as well as teachers exhibit difficulties identifying constant functions as functions (e.g., Carlson, 1998; Rasmussen, 2000); thus, there is a machine that acts as a constant function, i.e. every button produces the same color can. The purpose of this study is to determine the extent to which we were successful in designing for transformative learning (Mezirow, 2000; Taylor, 2007) related to the function definition. Our research question was:

> In what ways did PSMTs' personal definitions of function transform as a result of engaging with the vending machine task?

Methods

Participants and Data Sources

The participants in this study are 47 PSMTs enrolled in a secondary mathematics methods course at four different U.S. universities, ranging from five to 18 PSMTs at each university. The PSMTs were all undergraduate mathematics and/or mathematics education majors working toward earning their secondary mathematics teaching license. The individual degree programs all required at least 36 hours of mathematics, and these students had all successfully completed at least a second level calculus course at the time of the study. Every PSMT in the four methods courses took part in the study (N = 55). However, there were some PSMTs that did not have complete data sets these participants were removed, leaving 47 PSMTs in this particular study.

Data for this study consists of all of the PSMTs' work related to the Vending Machine task. This current paper represents a subset of a larger project and he subset of data relevant to this paper is written pre- and post-definitions of function on the Vending Machine task worksheet.

Analysis of Pre- and Post-definitions.

Pre- and post-definitions were entered into a spreadsheet for analysis. They were then coded using a codebook which was developed in a previous study and for which reliability was established (Author et al., 2018). All 94 definitions (47 pre and 47 post) were double-coded by two of the four authors using this codebook. To ensure reliability, coding was done in subsets of the data corpus, and coders compared codes, discussed, and resolved discrepancies (DeCuir-Gunby, Marshall, & McCulloch, 2011).

Similarly to Vinner and Dreyfus (1989), each definition was coded in terms of 1) accuracy, 2) focus, and 3) attention to output. In terms of *accuracy*, each definition was assigned a code of correct, incorrect, or close to correct. Key elements of a correct definition were 1) the definition was not limited to a specific type of function (e.g. linear or quadratic), or to a particular representation (e.g., equation), and 2) the definition addressed the idea that functions map each input to one and only one output, i.e., the univalence condition. Definitions coded as close to correct included those that indicated each input has one and only one output, but were not classified as correct because they were not general enough (e.g., the definition limited a function to a particular representation, such as an equation).

In terms of *focus*, each definition was coded regarding whether the definition indicated a function was a relationship (or mapping), an object, or neither. We referred to this set of codes as *focus*, as they indicated how the students "saw" function. We note that our use of the term *object* differs from its meaning in the APOS framework (Asiala et. al., 1996). In general, if a student identified a function with a representation or representations (e.g., "a function is an equation…"), then the definition was assigned a code of object. If the definition referred to a function as a relationship or mapping between variables or sets, it was coded as relationship. Finally, some definitions did not identify a function as an object or a relationship, but simply described some property of a function, e.g., "a function passes the vertical line test," then the definition was coded as neither. Although this code was intended to be mutually exclusive, there were a few definitions that identified a function as both a relationship and an object.

Finally, definitions were coded according to whether or not they *attended to output*. In order for a definition to be coded as attending to output, the definition needed to note something special or unique about the output. For example, "an equation with an input and an output" would not be considered as attending to output, while "an equation where each input has exactly one output" would. In addition, any definition which included mention of the vertical line test was coded as attending to output.

After coding was completed, results for each code were summarized and analyzed for patterns and themes that provided insight to transformations of students' conceptions related to the definition of function.

Results

PSMTs' Personal Definitions of Function

Given the applet design goal of disrupting students' meaning schemes for the concept of function, we noted how many students changed their definition from pre to post (students were given the option of not changing their definition from to pre to post as well). The number and percentage of definitions that were classified as correct, close to correct, or incorrect, pre- and post- engagement with the applet are shown in Table 1.

Table 1: Accuracy of Function Pre- and Post-Definitions

	Correct n (%)	Close to Correct n (%)	Incorrect (n (%)
Pre	4 (8.5%)	9 (19.1%)	34 (72.4%)
Post	7 (14.9%)	18 (38.3%)	22 (46.8%)

While 36 of the 47 PSMTs made a change to their definition, in many cases the post-definition did not change in terms of accuracy. Of those 36 that revised their definitions, 15 PSMTs improved the accuracy of their definition from pre to post, one PSMT's definition degenerated, and the rest of the definitions did not change with respect to accuracy. All 15 PSMTs whose definition improved started with incorrect definitions; three improved to a correct definition, and the other 12 moved from incorrect to close to correct. An example of a change for incorrect to correct is PSMT 17 who changed from "A function describes a relationship between 2 variable where the value of one variable determines the value of the other variable. A function must pass the vertical line test" to "A function describes a relationship between the domain and the range where for each input, or each value in the domain, there is only one corresponding output, or value in the range." With the PSMT noting "I changed this definition so that it focused on the number of outputs." An example of a change from incorrect to close to correct is PSMT 30 who went from "Function: an identity with more than one variable" to "A function is an equation that for every input (usually x) there is one output (usually y). It ceases being a function when multiple outputs exist for one input." The one PSMT whose definition declined with regard to accuracy went from close to correct to incorrect.

In terms of focus, the frequencies and percentage of definitions classified as relationship, object, both, or neither is depicted in Table 2. The notable result and very important with respect to focus is that while *object* was the most common code for the pre-definitions, *relationship* was the most common code for the post-definitions. This change corresponds both with the improvement in accuracy noted above.

Table 2: Focus of Function Pre- and Post-Definitions

Focus	Relationship n (%)	Object n (%)	Neither (n (%)	Both (n (%)
Pre	17 (36%)	19 (40%)	7 (15%)	4 (9%)
Post	20 (43%)	15 (32%)	9 (19%)	3 (6%)

Finally, the classification of attention to output had the most drastic change from pre- to post-definition. 60% (n= 28) attended to the output in their pre-definition and 89% (n= 42) attended to the output in their post definition. All of the 28 PSMTs who attended to output in their pre-definition continued to do so in their post-definition, and 14 of those who did not attend to the output in their pre-definition did so in their post-definition. Examples of PSMTs paying changing to pay attention to output are PSMT 12 who changed from “An expression involving more than one variable,” to “An expression involving more than one variable. For each input there is only one output” and PSMT 29 who moved from “A relationship that maps inputs and outputs and has some combination of variables and constants to “A relationship that maps every input to one output consistently.”

Overall the number of PSMTs who experienced a shift in accuracy of their personal definition, and who attended to output in their post-definition, and did not in their pre-definition indicates engagement with the applet resulted in transformations of their articulated conceptions of the definition function.

Discussion

The purpose of the design of the Vending Machine applet and this study was to elicit PSMTs’ personal definitions of functions and attempt to challenge and transform those definitions. The PSMTs exhibited many difficulties consistent with research literature on understanding of function suggesting that the applet design was effective in this regard. Furthermore, there is evidence that engaging with the vending machine applet resulted in most PSMTs reconsidering and refining their personal definitions of function in a positive direction.

Unsurprisingly, PSMTs’ struggled to articulate a complete definition of function, with much focus on objects rather than relationships or mappings as has been shown in previous research (e.g., Breidenbach et al., 1992; Carlson, 1998; Even, 1993). As future teachers of the function concept, perhaps the most concerning issue in PSMTs’ articulated definition, prior to engagement with the applet, was that 59% did not attend to the univalence requirement. However, as a result of engaging with the Vending Machine applet 89% attended to the univalence requirement in their post-definitions.

To enable and facilitate this change we drew on transformation theory (Mezirow, 2000), and Drijvers’s (2015) notion of the importance of didactical possibilities in the design of advanced digital technology applications, to guide the creation of an applet to trigger dilemmas that address common conceptions from the literature on distinguishing functions and non-functions. The use of advanced digital technology, allowed us to create a task with which the PSMTs could interact independently and which, with the immediate feedback of the machine outputs allowed them to formulate conjectures as they worked and test those conjectures without having to wait for a class discussion or intervention from an instructor. Our use of function machine, in the form of vending machines, as a cognitive root (Tall et al., 2000) proved to be accessible and meaningful for the PSMTs. One of the persistent problems noted in the literature is privileging algebraic representations (Even, 1990, 1993; Wilson, 1994) and putting PSMTs in the context of the vending machines appears to have mitigated this problem.

Conclusion

It is crucial that PSMTs have a solid understanding of function, know variations in the definition of function, develop the ability to translate among different representations of functions, and know when to use each definition based on context. This specialized content knowledge is needed to understand and plan for the diverse student conceptions they will encounter during instruction related to functions. While there is a vast literature base on the limited conceptions of functions PSMTs often develop through high school and undergraduate mathematics, little is known about how to transform them after years of building on them in algebraic contexts. The results of this study indicate that by removing PSMTs from familiar function contexts and designing to trigger dilemmas based on conceptions identified in the literature, we can transform PSMTs' personal definitions of function in a positive direction.

The results of this study suggest that coupling the research base in mathematics education with transformation theory to guide the design of learning experiences for PSMTs is promising, particularly in a technology-rich environment. Transformation theory values and leverages the wide-range of mathematical experiences of PSMTs, combining this with what research has revealed about learners' conceptions of a particular mathematical concept provides mathematics teacher educators a framework upon which to design such experiences. Given these findings, we hope that mathematics teacher educators will consider the use of transformative theory when designing learning experiences for PSMTs and inservice teachers.

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